DECOMPOSITION THEOREMS FOR JET BUNDLES OF A FIBER BUNDLE

ADELINA MANEA
Transilvania University of Brasov
Brasov
Romania
e-mail: amanea28@yahoo.com

Abstract

Let \((E, \pi, M)\) be a fiber bundle. We prove that the \(k\)-th jet bundle of \(M\) is a direct summand of the \(k\)-th jet bundle of \(E\), when they are pulled-back to the \(k\)-th jet manifold of bundle \(\pi\). The case of bundles over foliated manifolds is also studied.

1. Preliminaries

In this paper, we answer to a right question which arises in study of geometry of jet bundles: When the first jet bundle of the base space of a fiber bundle could be seen as a direct summand of the first jet bundle of the total space? This question can be extended to the \(k\)-th jet bundles and to the foliated case, too.

In this section, we introduce some notions about \(k\)-th order jets of a fiber bundle following [1, 6]. Let \((E, \pi, M)\) be a fiber bundle, where \(\pi : E \to M\) is a surjective submersion, \(M\) is an \(m\)-dimensional differentiable manifold and the fiber dimension is equal to \(n\). For a local
chart \((V, (x^i))\) in \(M\), the adapted coordinate system in \(\pi^{-1}(V) \subset E\) is \((x^i, y^\alpha)\), where \(i = \overline{1, m}\) and \(\alpha = \overline{1, n}\). We shall use the same notation \(x^i\) for the coordinate functions \(x^i\) from \(M\) and for \(x^i \circ \pi\) from the manifold \(E\). A local section of the bundle \(\pi\) in \(x \in M\) is a map \(\Phi : V \rightarrow E, x \in V \subset M\) such that \(\pi \circ \Phi = 1_V\). The set of all local sections of \(\pi\) in \(x\) is denoted by \(\Gamma_x(\pi)\). The local representation of a section is \((x^i, \Phi^\alpha)\), where \(\Phi^\alpha = y^\alpha \circ \Phi\).

Let \(k \geq 1\) be a positive integer. We say that two local sections \(\Phi, \Psi \in \Gamma_x(\pi)\) are \(k\)-equivalent at \(x\), if \(\Phi(x) = \Psi(x)\) and if in some adapted coordinate system \((x^i, y^\alpha)\) around \(\Phi(x)\), the maps \(\Phi^\alpha\) any \(\Psi^\alpha\) have the same derivatives at \(x\) up to \(k\)-order, for every \(\alpha = \overline{1, n}\). The equivalence class containing \(\Phi\) is called the \(k\)-jet of the section \(\Phi\) at \(x\) and is denoted by \(J^k_x \Phi\).

Let \(I\) be the multiindex \((1 \leq i_1 \leq \ldots \leq i_r \leq m)\), with \(r = |I|\) the length of \(I\). We denote the derivation of \(r\)-order with respect to \(x^{i_1}, x^{i_2}, \ldots, x^{i_r}\) by

\[
\frac{\partial^{|I|}}{\partial x^{|I|}}.
\]

The \(k\)-th jet manifold of \(\pi\) is the set \(J^k \pi = \{j^k_x \Phi \mid x \in M, \Phi \in \Gamma_x(\pi)\}\).

It is a manifold with local charts \((U^k, u^k)\), where \(U^k = \{j^k_x \Phi \in J^k \pi \mid \Phi(x) \in U\}, (U, (x^i, y^\alpha))\) is a local chart on \(E\), and the functions \(u^k = (x^i, y^\alpha, y^\alpha_{\mid 1 \leq |I| \leq k})\), are defined by

\[
x^i(j^k_x \Phi) = x^i(x), \quad y^\alpha(j^k_x \Phi) = \Phi^\alpha(x), \quad y^\alpha_{\mid 1 \leq |I| \leq k} = \frac{\partial^{|I|} \Phi^\alpha}{\partial x^{|I|}}(x).
\]

Moreover, \((J^k \pi, \pi_k, M)\) and \((J^k \pi, \pi_{k,0}, E)\) are bundles with the surjection submersions.
\( \pi_k : J^k \pi \to M, \quad \pi_{k,0} : J^k \pi \to E, \) (2)

defined by \( \pi_k(j^k_x \Phi) = x \) and \( \pi_{k,0}(j^k_x \Phi) = \Phi(x) \), respectively. We also have the equality \( \pi \circ \pi_{k,0} = \pi_k \).

If \( 1 \leq l \leq k \), then the \( l \)-jet projection is the map

\[
\pi_{k,l} : J^k \pi \to J^l \pi, \quad \pi_{k,l}(j^k_x \Phi) = j^l_x \Phi.
\] (3)

A particular case of bundle is the trivial bundle \((M \times \mathbb{R}, pr_1, M)\).

The set of global sections of this bundle is the space of \( \mathcal{C}^\infty \)-differentiable functions on \( M \) and we denote it by \( \Omega^0(M) \). The set of local sections defined on a domain containing \( x \in M \) is denoted by \( \Omega^0_x(M) \). The first jet manifold of \( M \) is isomorphic to \( T^*M \times \mathbb{R} \). The \( k \)-th jet manifold of this trivial bundle is called the \( k \)-th jet manifold of \( M \)

\[
J^k M = \{ j^k_x f \mid x \in M, f \in \Omega^0_x(M) \}.
\]

Local coordinates of this manifold are \((x^i, \omega_\alpha)_{1 \leq |\alpha| \leq k}\), with

\[
\omega(j^k_x f) = pr_2(f(x)), \quad \omega_\alpha(j^k_x f) = \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x).
\]

The \( k \)-th jet manifold of \( E \) is the set \( J^k E = \{ j^k_a F \mid a \in E, F \in \Omega^0_x(E) \} \). Considering the multiindex \( I \) be \((i_1 \leq \ldots \leq i_r)\) and \( \Lambda \) be \((\alpha_1 \leq \ldots \leq \alpha_r)\), we have that the local coordinates in \( J^k E \) are

\[
(x^i, y^\alpha, z, z_I, z_\Lambda, z_{I\Lambda} \mid 1 \leq |I| \leq k, 1 \leq |\Lambda| \leq k, 1 \leq |I\Lambda| \leq k),
\]

respectively, with \( z(j^k_a F) = pr_2(F(a)) \) and

\[
z_I(j^k_a F) = \frac{\partial^{|I|} F}{\partial x^I}(a), \quad z_\Lambda(j^k_a F) = \frac{\partial^{|\Lambda|} F}{\partial y^\Lambda}(a),
\]
\[ z_{IA}(J^k \alpha) = \frac{\partial |I|}{\partial x^i \partial y^j}(a). \]

There are some facts that suggests the existence of a relation between bundles \((J^k M, \pi^M_k, M)\) and \((J^k E, \pi^E_k, E)\). These facts are [6]:

First: The bundle \((J^1 M, \pi^M_1, M)\) is isomorphic with the bundle \((T^* M \times \mathbb{R}, \tau^* M \times p_1, M)\).

Second: The pull-back bundle \((\pi^*(T^* M), \pi^*(\tau^*_M), E)\) of the cotangent bundle of \(M\) by \(\pi\) is a distinguished subbundle of \((T^* E, \tau^*_E, E)\), but it do not has a distinguished complement in the absence of a connection on \(\pi\). But, when these bundles are pulled-back to \(J^1 \pi\), then there is such a complement called the bundle of contact cotangent vectors. An element \((\eta, j^1_\pi \phi)\) of the pull-back bundle

\[ \pi^*_1(0,1) = \{(\eta, j^1_\pi \Phi) \in T^* E \times J^1 \pi, \quad \tau^*_E(\eta) = \pi^*_1(j^1_\pi \Phi) = \Phi(x)\} \]

is called a contact cotangent vector if \(\Phi^*(\eta) = 0\). We denote by \(\pi^*_1(\ker \Phi^*)\) the set of cotangent vectors in \((\pi^*_1(0,1))^j_\pi \Phi\) and the bundle \(C^* \pi^*_1(0,1)\) of contact cotangent vectors is the union of the fibers \(\pi^*_1(\ker \Phi^*)\), for \(j^1_\pi \Phi \in J^1 \pi\).

2. Decompositions of the \(k\)-th Jet Bundles of the Total Space of a Bundle

In this section, we prove that

**Theorem 2.1.** The pull-back bundle \((\pi^*_k(J^k M), \pi^*_k(\pi^M_k), J^k \pi)\) is a direct summand of the pull-back bundle \((\pi^*_k(0,1))^j_\pi \Phi, \pi^*_k(0,1), J^k \pi\).
Proof. We consider the pull-back bundles
\[ \pi_{k}^{*}(J^{k}M) = \{(j_{k}^{f}, j_{x}^{k}\Phi) \in J^{k}M \times J^{k}\pi, f \in \Omega^{0}(U_{x}), \Phi \in \Gamma_{x}(\pi)\}, \]
\[ \pi_{k,0}^{*}(J^{k}E) = \{(j_{a}^{k}F, j_{x}^{k}\Phi) \in J^{k}E \times J^{k}\pi, F \in \Omega^{0}(\pi^{-1}(U_{x})), \Phi \in \Gamma_{x}(\pi), a = \Phi(x)\}. \]

For every fixed point \( j^{k}\Phi \in J^{k}\pi \), we define the maps
\[ \xi_{j^{k}\Phi}^{i} : (\pi_{k,0}^{*}(J^{k}E))_{j^{k}\Phi}^{i} \to (\pi_{k}^{*}(J^{1}M))_{j^{k}\Phi}, \quad \xi_{j^{k}\Phi}^{i} = j_{\pi(a)}^{k}(F \circ \Phi), \]
\[ \zeta_{j^{k}\Phi}^{i} : (\pi_{k}^{*}(J^{k}M))_{j^{k}\Phi} \to (\pi_{k,0}^{*}(J^{k}E))_{j^{k}\Phi}, \quad \zeta_{j^{k}\Phi}^{i} = j_{\Phi(x)}^{k}(f \circ \pi|_{\pi^{-1}(U_{x})}). \]

It is straightforward to check that these maps are well-defined, since \( \pi(a) = x \) by definition, and the local representation \( \left( \frac{\partial|\mathcal{I}|(F \circ \Phi)}{\partial x^{I}}(x) \right) \) of \( j_{\pi(a)}^{k}(F \circ \Phi) \) depends only by the derivatives of \( F \) at \( a = \Phi(x) \), up to \( k \)-th order.

Similarly, \( j_{\Phi(x)}^{k}(f \circ \pi) \) has the local representation
\[ \left( \frac{\partial|\mathcal{I}|(f \circ \pi)}{\partial x^{I}}(\Phi(x)), \quad \frac{\partial|\mathcal{I}|(f \circ \pi)}{\partial y^{\lambda}}(\Phi(x)), \quad \frac{\partial|\mathcal{I}|+|\Lambda|(f \circ \pi)}{\partial x^{I}\partial y^{\lambda}}(\Phi(x)), \right) \]
for all \( 1 \leq |\mathcal{I}| \leq k, 1 \leq |\Lambda| \leq k, \) and \( 1 \leq |\mathcal{I}|+|\Lambda| \leq k \), which depends only by the derivatives of \( f \) at \( x \) up to \( k \)-th order. Moreover, since \( \pi \circ \Phi = 1_{U_{x}} \), we have the equality
\[ \xi_{j^{k}\Phi}^{i} \circ \zeta_{j^{k}\Phi}^{i} = 1_{\pi_{k}^{*}(J^{k}M)}(j_{j^{k}\Phi}^{k}). \] (4)

Hence, we obtain the maps
\[ \xi : \pi_{k,0}^{*}(J^{k}E) \to \pi_{k}^{*}(J^{k}M), \quad \xi(j_{a}^{k}F, j_{x}^{k}\Phi) = (\xi_{j^{k}\Phi}^{i}(j_{a}^{k}F), j_{x}^{k}\Phi), \]
\[ \zeta : \pi_{k}^{*}(J^{k}M) \to \pi_{k,0}^{*}(J^{k}E), \quad \zeta(j_{a}^{k}F, j_{x}^{k}\Phi) = (\zeta_{j^{k}\Phi}^{i}(j_{a}^{k}F), j_{x}^{k}\Phi). \]
We also have \( \xi \circ \zeta = 1_{\pi_k^*(J^kM)} \), so \( \xi \) is surjective.

It follows that the exact sequence of bundles

\[
0 \to \text{Ker}\xi \to \pi_{k,0}^*(J^kE) \xrightarrow{\xi} \pi_k^*(J^kM) \to 0
\]

is splitting by \( \zeta \). It follows that \( \zeta \circ \xi \) is a projection, and the bundle \( \pi_k^*(J^kM) \) is isomorphic with a direct summand of \( \pi_{k,0}^*(J^kE) \). In the case of \( k = 1 \), the complement of \( \pi_1^*(J^1M) \) is

\[
\text{Ker}\xi = \{(j_0^1F, j_0^1\Phi) \in \pi_{1,0}^*(J^1E), (d_aF, j_x^1\Phi) \in C^*\pi_{1,0}\}.
\]

Indeed, the conditions \( \frac{\partial(F \circ \Phi)}{\partial x^i} = 0 \) for all \( i \) are equivalent to \( \Phi^*(d_aF) = 0 \).

3. Case of Bundles over Foliated Manifolds

We introduce some notions about foliated manifolds following [4, 8]. Let \((M, \mathcal{F})\) be a \( p \)-dimensional foliated manifold and \( (x^i) = (x^a, x^u) \) adapted coordinate system on the open set \( V \subset M \), where \( a = 1, m - p, u = m - p + 1, m \), such that the points in the same leaf \( \mathcal{L} \cap V \) have their first \( m - p \) coordinates equal, and are distinguished by their last \( p \) coordinates.

In the following, the indices will take the following values:
\( i, j, ... = 1, m; a, b, ... = 1, m - p; u, v, ... = m - p + 1, m; \) and \( \alpha, \beta, ... = 1, n \).

Some geometrical properties of bundles over foliated manifolds was investigate in [5]. Let \((E, \pi, M)\) be a fiber bundle over the foliated manifold \((M, \mathcal{F})\) and \( \mathcal{F}' \) be the pull-back foliation of \( \mathcal{F} \) by \( \pi \). The leaves of \( \mathcal{F}' \) are connected components of the inverse images by \( \pi \) of the leaves of \( \mathcal{F} \) [4]. We denote by \((T\mathcal{F}, \tau_F, M)\) and \((T\mathcal{F}', \tau_{F'}, E)\) the structural bundles of \( \mathcal{F} \) and \( \mathcal{F}' \), respectively.
Remark 3.1. Let \( \Phi \) be a section of \( \pi \). For any \( \zeta \in TF \), we have \( \pi_*(\Phi_*(\zeta)) = \zeta \in TF \), so \( \Phi_*(\zeta) \in TF' \). It can say that every section on \( \pi \) is a foliated map between the manifolds \((M, F)\) and \((E, F')\).

In the following, we extend the notions about first order leafwise jets from [2] to the \( k \)-th order jets. For a local chart \((V, (x^a, x^u))\) in \( M \), we have an adapted local chart \((U, (x^a, x^u, y^a))\) in \( E \).

**Definition 3.1.** We say that two local sections \( \Phi, \Psi \in \Gamma_x(\pi) \) are *leafwise \( k \)-equivalent* at \( x \in M \) if \( \Phi(x) = \Psi(x) \) and if, in some adapted coordinate system \((x^a, x^u, y^a)\) around \( \Phi(x) \)

\[
\frac{\partial|\Phi_{\alpha}^a}{\partial x_I^I}(x) = \frac{\partial|\Psi_{\alpha}^a}{\partial x_I^I}(x),
\]

for every multiindex \( I = (u_1 \leq u_2 \leq \ldots \leq u_r) \), \( u_i = m - p + 1, m \) for all \( i = 1, r \) and \( 1 \leq r \leq k \). The equivalence class containing \( \Phi \) is called the *leafwise \( k \)-jet* of \( \Phi \) and is denoted by \( j_x^l, k \Phi \).

**Remark 3.2.** By definition, if \( j_x^l \Phi = j_x^l \Psi \), then \( j_x^l, k \Phi = j_x^l, k \Psi \). The reverse is not true: Indeed, in the case of vector bundles, \( j_x^l, k \Phi = j_x^l, k (\Phi + \Omega) \) for every local basic section \( \Omega \in \Gamma_x(\pi) \) (that means \( \frac{\partial(y^u \circ \Omega)}{\partial x^u}(x) = 0 \)), but \( j_x^1 \Phi \neq j_x^1 (\Phi + \Omega) \), hence \( j_x^l, k \Phi \neq j_x^k (\Phi + \Omega) \).

**Remark 3.3.** We also can say that two local sections \( \Phi, \Psi \in \Gamma_x(\pi) \) are leafwise \( k \)-equivalent at \( x \in M \), if they are \( k \)-equivalent at \( x \) in the leaf through \( x \) in the foliated manifold \( M \).

The \( k \)-th leafwise jet manifold of \( \pi \) is the set \( J^{l,k} \pi = \{ j_x^{l,k} \Phi \mid x \in M, \Phi \in \Gamma_x(\pi) \} \).

Let \( \mathcal{A}_E = \{(U, u = (x^a, x^u, y^a))\} \) be an adapted atlas on \( E \). The
induced coordinate system \( (U^{l,k}, u^{l,k}) \) on the set \( J^{l,k} \) is defined by

\[
U^{l,k} = \left\{ j_x^{l,k} \Phi \in J^{l,k} \mid \Phi(x) \in U \right\},
\]

\[
u^{l,k} = (x^a, x^u, \omega^a, \omega_I^u),
\]
(6)
given by \( x^a(j_x^{l,k} \Phi) = x^a(x), x^u(j_x^{l,k} \Phi) = x^u(x), \omega^a(j_x^{l,k} \Phi) = \Phi^a(x), \omega_I^u(j_x^{l,k} \Phi) = \frac{\partial \Phi^a}{\partial x^I}(x) \) for every multiindex \( I = (u_1 \leq u_2 \leq \ldots \leq u_r) \) and \( 1 \leq |I| \leq k \).

It is easy to verify that the collection of all charts \( (U^{l,k}, u^{l,k}) \) is an atlas on \( J^{l,k} \).

Moreover, the maps

\[
\pi_k^l : J^{l,k} \to M; \quad \pi_k^l(j_x^{l,k} \Phi) = x,
\]
\[
\pi_{k,0}^l : J^{l,k} \to E; \quad \pi_{k,0}^l(j_x^{l,k} \Phi) = \Phi(x),
\]
for every \( j_x^{l,k} \Phi \in J^{l,k} \) are surjective submersions corresponding to \( \pi_k, \pi_{k,0} \) from the previous section. The Remark 3.2 assures that the map

\[
\psi_k^l : J^k \to J^{l,k}; \quad \psi_k^l(j_x^k \Phi) = j_x^{l,k} \Phi,
\]
(7)
is well-defined. Moreover, the following relations hold

\[
\pi_{k,0}^l \circ \psi_k^l = \pi_k,0; \quad \pi_k^l \circ \psi_k^l = \pi_k; \quad \pi_k \circ \pi_{k,0}^l = \pi_k^l.
\]
(8)

Considering now the trivial bundles, we obtain the \( k \)-th leafwise jet manifolds of manifolds \( (M, \mathcal{F}) \) and \( (E, \mathcal{F}') \), respectively,

\[
J^{l,k} M = \left\{ j_x^{l,k} f \mid x \in M, f \in \Omega^0_x(M) \right\};
\]
\[
J^{l,k} E = \left\{ j_u^{l,k} F \mid \alpha \in E, F \in \Omega^0_x(E) \right\}.
\]
The local coordinates on these manifolds are \( (x^a, x^u, \omega, \omega_I) \) and \( (x^a, x^u) \),
\( y^\alpha, z, z_I, z_{IA} \) with local functions \( \omega(j^{l,k}_x f) = pr_2(f(x)), \omega_I(j^{l,k}_x f) = \frac{\partial^{|I|} f}{\partial x^I}(x), z_I(j^{l,k}_a F) = pr_2(F(a)), z_{IA}(j^{l,k}_a F) = \frac{\partial^{|I|} |A| F}{\partial y^I}(a), \) and \( z_{IA}(j^{l,k}_a F) = \frac{\partial^{|I|} |A| F}{\partial x^I \partial y^A}(a), \) for every multiindices \( I = (u_1 \leq \ldots \leq u_r), \) and \( A \) be \( (\alpha_1 \leq \ldots \leq \alpha_d), \) and \( 1 \leq |I| \leq k, 1 \leq |A| \leq k, 1 \leq |A| + |I| \leq k. \) We also have the surjections

\[ \pi^{l,M}_k : J^{l,k} M \to M, \quad \pi^{l,M}_k(j^{l,k}_x f) = x, \]
\[ \pi^{l,E}_k : J^{l,k} E \to E, \quad \pi^{l,E}_k(j^{l,k}_a F) = a, \]
and corresponding to map (7), we have the following surjections:

\[ \psi^{l,k}_M : J^k M \to J^{l,k} M, \quad \psi^{l,k}_E : J^k E \to J^{l,k} E, \]
\[ \psi^{l,k}_M(j^k_x f) = j^{l,k}_x f, \quad \psi^{l,k}_E(j^k_a F) = j^{l,k}_a F. \] (9)

We can give the following result, similar with Theorem 2.1:

**Theorem 3.1.** The pull-back bundle \( (\pi^{l,*}_k(J^{l,k} M), \pi^{l,*}_k(J^{l,k} M), J^{l,k} \pi) \)

is a direct summand of the pull-back bundle \( (\pi^{l,*}_{k,0}(J^{l,k} E), \pi^{l,*}_{k,0}(J^{l,k} E), J^{l,k} \pi). \)

For the pull-back bundles

\[ \pi^{l,*}_k(J^{l,k} M) = \{(j^{l,k}_x f, j^{l,k}_x \Phi) \in J^{l,k} M \times J^{l,k} \pi, \]
\[ f \in \Omega^0(\overline{U}_x), \Phi \in \Gamma_x(\pi), \}
\[ \pi^{l,*}_{k,0}(J^{l,k} E) = \{(j^{l,k}_a F, j^{l,k}_x \Phi) \in J^{l,k} E \times J^{l,k} \pi, \]
\[ F \in \Omega^0(\overline{\pi}^{-1}(\overline{U}_x)), \Phi \in \Gamma_x(\pi), a = \Phi(x), \}

we define the maps

\[ \xi^{l,k}_{j^{l,k}_x \Phi} : (\pi^{l,*}_{k,0}(J^{l,k} E))_{j^{l,k}_x \Phi} \to (\pi^{l,*}_k(J^{l,k} M))_{j^{l,k}_x \Phi}. \]
for every fixed point \( j^l_k \Phi \in J^l_k \pi \). These maps are well-defined because every \((j^l_k F, j^l_k \Phi) \in \pi^l_{k, 0}(J^l_k E)\) satisfies \( \alpha = \Phi(x) \), which implies \( \pi(\alpha) = x \), and because they do not depend by the representants of \( j^l_k F \) and \( j^l_k f \), respectively. In calculation, a key fact is that surjection \( \pi \) is a foliated map between \((M, \mathcal{F})\) and \((E, \mathcal{F}')\), since the leaves of \( \mathcal{F}' \) are inverse images of the leaves of \( \mathcal{F} \) by \( \pi \). Locally, if \((\pi^a, \pi^u)\) is the local representation of \( \pi \), then we have \( \frac{\partial \pi^a}{\partial \xi} = 0 \) and \( \frac{\partial \pi^a}{\partial y^a} = 0 \). Moreover, since \( \pi \circ \Phi = 1_{\xi_u} \), we have the equality

\[
\xi^l_{j^l_k \Phi} \circ \xi^l_{j^l_k \Phi} = 1_{\pi^l_{k, 0}(J^l_k M)_{j^l_k \Phi}}. \tag{10}
\]

Hence, we have the maps

\[
\xi^l_{j^l_k \Phi} : \pi^l_{k, 0}(J^l_k E) \rightarrow \pi^l_{k, 0}(J^l_k M),
\]

\[
\xi^l_{j^l_k \Phi} : \pi^l_{k, 0}(j^l_k F, j^l_k \Phi) = (\xi^l_{j^l_k \Phi}(j^l_k F), j^l_k \Phi),
\]

\[
\xi^l_{j^l_k \Phi} : \pi^l_{k, 0}(J^l_k M) \rightarrow \pi^l_{k, 0}(J^l_k E),
\]

\[
\xi^l_{j^l_k \Phi} : \pi^l_{k, 0}(j^l_k f, j^l_k \Phi) = (\xi^l_{j^l_k \Phi}(j^l_k f), j^l_k \Phi),
\]

with \( \xi^l_{j^l_k \Phi} = 1_{\pi^l_{k, 0}(J^l_k M)} \), so \( \xi^l_{j^l_k \Phi} \) is surjective. The exact sequence of bundles
is splitting by $\zeta^{l,k}$. It follows that $\zeta^{l,k} \circ \zeta^{l,k}$ is a projection, and the bundle $\pi_{k,0}^{l,*}(J^{l,k}M)$ is isomorphic with a direct summand of $\pi_{k,0}^{l,*}(J^{l,k}E)$. Its complement is $\ker \zeta^{l,k}$.

For the case $k = 1$, we can describe this complement as it follows. In [3], we give the following definition:

**Definition 3.2.** Let $(E, \pi, M)$ be a bundle over the foliated manifold $M$ and $x \in M$, $\Phi \in \Gamma_{x}(\pi)$, $\zeta \in T_{x}\mathcal{F}$. The leafwise holonomic lift of $\zeta$ by $\Phi$ is defined to be

$$\left(\Phi_{*}(\zeta), j^{1,1}_{x}\Phi\right) \in \pi_{1,0}^{l,*}(T\mathcal{F}).$$

We denote by $\Phi_{*}(T_{x}\mathcal{F})$ the collection of holonomic lift of tangent vector from $T_{x}\mathcal{F}$ by $\Phi$. Then, we have the following proposition:

**Proposition 3.1.**

$$\ker \zeta^{l,1} = \{(j^{1,1}_{a}F, j^{1,1}_{x}\Phi) \in \pi_{1,0}^{l,*}(J^{l,1}E), d_{a}F(X) = 0, (\forall)X \in \Phi_{*}(T_{x}\mathcal{F})\}.$$

**Proof.** Indeed, $(j^{1,1}_{a}F, j^{1,1}_{x}\Phi) \in \ker \zeta^{l,1}$ implies $\frac{\partial(F \circ \Phi)}{\partial x^{u}} = 0$ in every local chart around $a = \Phi(x)$. That means

$$\frac{\partial F}{\partial x^{u}}(\Phi(x)) + y^{u}_{a}(j^{1,1}_{x}\Phi) \cdot \frac{\partial F}{\partial y^{a}}(\Phi(x)) = 0. \quad (11)$$

In [3], we found the coordinate representation of a leafwise holonomic lift of a tangent vector as it follows. Supposing that

$$\zeta = \xi^{u} \frac{\partial}{\partial x^{u}} \bigg|_{x} \in T_{x}\mathcal{F},$$
we have

\[ \Phi_*(\zeta) = \zeta^u \Phi_* \left( \frac{\partial}{\partial x^u} \bigg|_x \right) = \zeta^u \left( \frac{\partial}{\partial x^u} |_{\Phi(x)} + y_\alpha^a (J^1_x \Phi) \frac{\partial}{\partial y^a} |_{\Phi(x)} \right) . \]

Hence, by (10), it follows \( \Phi_*(\zeta)(F) = 0 \), for every \( \zeta \in T_x F \), which ends the proof.

Finally, we remark that the following diagram is commutative:

\[ \pi^E_{k,0} (J^k E) \quad \pi^E_{k,1} \times \pi^M_{k,1} \quad \pi^I_{1,0} (J^1 E) \quad \pi^I_{1,1} (J^l,1 E) \]

\[ \pi^M_{k,0} (J^k M) \quad \pi^I_{1} (J^1 M) \quad \pi^I_{1,1} (J^l,1 E) \]

where the maps \( \pi^E_{k,1} \) and \( \pi^M_{k,1} \) are corresponding to \( \pi_{k,l} \) from (3), for the case of trivial bundles and \( l = 1 \).

**References**


